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APPROXIMATE FINITE ELEMENT MODELS FOR STRUCTURAL CONTROL

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Abstract

Approximate finite element models are developed for the purpose of preserving the tridiagonality of the mass and stiffness matrices in the state space model matrices. These approximate models are utilized in the design of active structural control laws for large flexible structures.

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Introduction

In this paper we consider the problem of approximating a class of linear matrix-second-order systems which often result from applying finite element methods to discretize structural dynamics problems. These finite element models are of the type:

$$E_i\ddot{x}_{i-1} + D_i\ddot{x}_i + F_i\ddot{x}_{i+1} = B_ix_{i-1} + A_ix_i + C_ix_{i+1} + f_i \quad , \qquad i = 1, \ldots, n \tag{1}$$

where $E_1 = B_1 = F_n = C_n = 0$. This type of system is characterized by two block tridiagonal matrices: the mass matrix M which is multiplied in (1) to an acceleration vector, $\ddot{x} = (\ddot{x}_1, \ldots, \ddot{x}_n)^T$, and the stiffness matrix K, which is multiplied to a displacement vector, $x = (x_1, \ldots, x_n)^T$, *i.e.*

$$M = \begin{bmatrix} D_1 & F_1 & 0 & \cdots & & 0 \\ E_2 & D_2 & F_2 & \ddots & & \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & E_{n-1} & D_{n-1} & F_{n-1} \\ 0 & \cdots & & 0 & E_n & D_n \end{bmatrix} , \tag{2}$$

$$K = -\begin{bmatrix} A_1 & C_1 & 0 & \cdots & & 0 \\ B_2 & A_2 & C_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & B_{n-1} & A_{n-1} & C_{n-1} \\ 0 & \cdots & & 0 & B_n & A_n \end{bmatrix} . \tag{3}$$

Motivations for finding approximate finite element models are derived from the need to obtain state space models which preserve the bandedness in the mass and stiffness matrices. Assuming that M is invertible, the system (1) is expressible in the form:

$$\ddot{x} = -M^{-1}Kx + M^{-1}f \quad , \tag{4}$$

where $f = (f_1, \ldots, f_n)^T$. Since the inverse of M need not be a diagonal matrix, therefore, the block tridiagonality of K is not preserved in the matrix product $M^{-1}K$. The bandedness of this matrix product is preserved only if the mass matrix is a diagonal matrix, *i.e.*, $E_i = F_i = 0$, for $i = 1, \ldots, n$.

In finite element methods, it is known that if a Rayleigh-Ritz procedure is used in generating the discretized equations, the mass matrix M is generally a banded matrix, such as the block tridiagonal matrix exhibited in (2), but not a diagonal matrix. For the purpose of increasing computational efficiency, eigenvalue problems associated with structural analysis are often solved not with the banded mass matrix, but with an alternate diagonal mass matrix which is derived from lumping the mass at the nodes. This mass lumping practice, which essentially produces a diagonal mass matrix, preserves the bandedness (or block tridiagonality) of the stiffness matrix, thus contributing to the efficiency of the eigenvalue computation.

Lumping the mass, however, can lead to a serious loss of accuracy in the approximation of the eigenvalues of the continuum system. This drawback was first reported in [1], and discussed subsequently in the finite element textbook by Strang and Fix [2]. In spite of its shortcomings, the lumped mass approach was nevertheless recommended in [3], in dealing with systems with a large number of nodes. The rationale provided therein was that the additional computational resource imposed by using the Rayleigh-Ritz procedure may well be more profitably spent on refining the finite element mesh, yet adhering to a lumped mass approach. The problem of developing approximate finite element models lies in determining the conditions under, and the degree to which the matrix product $M^{-1}K$ can be be approximated by a block tridiagonal matrix.

The outline of this paper is as follows. First a lemma concerning the inversion of a class of block tridiagonal matrices is presented. This lemma utilizes an $L-U$ factorization of the matrix and presents a closed form formula for the inverse. The main result is contained in Theorem 1 in which a family of approximate finite element models is developed for (1). The developed approximate models are utilized in an active structural control design in which the block tridiagonality of the state space matrices is exploited. Theorem 2 shows the degree of suboptimality achievable with this control design. A truss structure is used to illustrate the development of the approximate models and the active structural control design.

Inversion of block tridiagonal matrices

For the class of linear matrix-second-order systems given by (1), the matrix M is assumed to be invertible. Moreover, the diagonal submatrices D_i of M are also invertible. The inversion of M poses no particular difficulties even when a closed form expression for M^{-1} is required. The inversion process nevertheless destroys the block tridiagonality of the matrix. The matrix M^{-1} would be at best approximately block tridiagonal. The notion of approximate block tridiagonality is introduced as follows.

Definition 1 (Approximate Block Tridiagonality):

An invertible matrix P is approximate block tridiagonal if there exists a block tridiagonal matrix P_0 such that

$$P = P_0 + P_1 \tag{5a}$$

and

$$0 < \|P_0\|^{-1}\, \|P_1\| < 1 \quad , \tag{5b}$$

for some matrix norm $\|\cdot\|$.

In this section, we show that the class of matrices having approximate block tridiagonal inverses are block tridiagonal matrices which satisfy a block diagonal dominance condition. Diagonal dominance is a well established concept in matrix theory [4]. In this paper, we use exclusively the following definition for block diagonal dominance.

Definition 2 (Implicit Block Diagonal Dominance):

Let P be a matrix which has k block rows and k block columns, and P_{ij} be the ij^{th} submatrix of P. P is implicit block *column* diagonal dominant if for some matrix $\|\cdot\|$,

$$\sum_{\substack{j=1\\ j\neq i}}^{k} \|P_{ji}P_{ii}^{-1}\| < 1 \quad , \quad i = 1, \ldots, k \quad . \tag{6}$$

Another possible definition which will *not* be adopted herein is:

Definition 3 (Explicit Block Diagonal Dominance):

The matrix P is explicit block diagonal dominant if

$$\|P_{ii}^{-1}\| \sum_{\substack{j=1 \\ j\neq i}}^{k} \|P_{ij}\| < 1 \quad , \quad i = 1, \ldots, k \quad . \tag{7}$$

The explicit diagonal dominance condition is more exclusive than the implicit one since (6) implies (7), yet the converse is not true.

Let

$$a_i = \|F_{i-1}D_i^{-1}\| \quad , \quad i = 2, \ldots, n \tag{8}$$

$$b_j = \|E_{j+1}D_j^{-1}\| \quad , \quad j = 1, \ldots, n-1 \quad . \tag{9}$$

Following Definition 2, the matrix M is block diagonal dominant if

$$a_k + b_k < 1 \quad , \quad k = 1, \ldots, n \quad , \tag{10}$$

with $a_1 = b_n = 0$.

Lemma 1:

If the matrix in (2) is block diagonal dominant, then there exists a block $L - U$ factorization

$$M = \underbrace{\begin{bmatrix} I & 0 & & \cdots & 0 \\ L_2 & I & & & \\ 0 & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & \ddots & \\ & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & L_n & I \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} U_1 & F_1 & 0 & \cdots & 0 \\ 0 & U_2 & & & \vdots \\ \vdots & \ddots & & \ddots & 0 \\ & & \ddots & \ddots & F_{n-1} \\ 0 & \cdots & & 0 & U_n \end{bmatrix}}_{\mathcal{U}} , \tag{11}$$

furthermore,

$$\|L_i\| \leq \tilde{b}_{i-1} < 1 \quad , \quad i = 2, \ldots, n \tag{12}$$

where

$$\tilde{b}_1 = b_1 \quad , \qquad \tilde{b}_j = \frac{b_j}{1 - \tilde{b}_{j-1}a_j} \quad . \tag{13}$$

The proof of this lemma is given in the Appendix.

If in stead of implicit block diagonal dominance, we have an explicit block diagonal dominance condition placed on the mass matrix M, the above Lemma has been proved in [5, Theorem 5.5-1]. However, we found that such a condition is too restrictive for the finite element models that we have examined. Using the $L - U$ factorization obtained in the above lemmas, the inversion of the mass matrix is relatively straightforward since it is reduced to the inversion of the $L - U$ factors which only requires the inversion of upper and lower block triangular matrices. The next lemma deals with the inversion of the "$\mathcal{L}$ factor" in (11).

Lemma 2:

The inverse of $\mathcal{L}$ is also a lower block triangular matrix which has a main diagonal block of identity matrices; its j^{th} subdiagonal block, *i.e.*, the j^{th} diagonal block below the main diagonal block is given by

$$(-1)^j L_{j+1}L_jL_{j-1}\dots L_2, (-1)^j L_{j+2}L_{j+1}L_jL_{j-1}\dots L_3, \dots, (-1)^j L_nL_{n-1}\dots L_{n-j+1}. \tag{14}$$

There are $n-1$ subdiagonal blocks and $n-j$ block elements in the j^{th} subdiagonal block. Each of the block matrices in the j^{th} subdiagonal block is a matrix product of j of the matrices L_i, $i = 1, \dots, n$.

The proof involves the application of a matrix inversion formula [5] repetitively. The structure of $\mathcal{L}^{-1}$ can be conveniently illustrated with small n, e.g., $n = 4$, in which case,

$$\mathcal{L}^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ -L_2 & I & 0 & 0 \\ L_3L_2 & -L_3 & I & 0 \\ -L_4L_3L_2 & L_4L_3 & -L_4 & I \end{bmatrix} , \tag{15}$$

The inversion of the "$\mathcal{U}$ factor" in (11) utilizes Lemma 2 through the observation that

$$\mathcal{U}^{-1} = \mathrm{diag}\,(U_1^{-1}, U_2^{-1}, \dots, U_n^{-1})\,\mathcal{G}^{-1} \tag{16}$$

where

$$\mathcal{G} \triangleq \begin{bmatrix} I & G_1 & 0 & \dots & 0 \\ 0 & I & G_2 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & G_{n-1} \\ 0 & & \dots & 0 & I \end{bmatrix} , \tag{17}$$

$$G_i \triangleq F_iU_{i+1}^{-1} \quad , \quad i = 1, \dots, n-1 \quad . \tag{18}$$

The following corollary results from applying Lemma 2 to (16).

Corollary 1:

The inverse of $\mathcal{U}$ in (11) is an upper block triangular matrix with a main diagonal block of identity matrices. Its i^{th} superdiagonal block, *i.e.*, the i^{th} diagonal block above the main diagonal block, is given by

$$(-1)^j U_1^{-1}G_1G_2\dots G_j, (-1)^j U_2^{-1}G_2G_3\dots G_{j+1}, \dots,$$
$$(-1)^j U_j^{-1}G_{n-j}G_{n-j+1}\dots G_{n-1} \quad . \tag{19}$$

There are $n-1$ superdiagonal blocks and $n-i$ block elements in the i^{th} superdiagonal block. The k^{th} block element in the i^{th} superdiagonal block is a product between U_k^{-1} and i of the matrices G_j, $j = 1, \dots, n-1$.

As an illustration of the structure of $\mathcal{U}^{-1}$, consider again a mass matrix partitioned into four block rows and columns, *i.e.*, $n = 4$.

In this case,

$$\mathcal{U}^{-1} = \begin{bmatrix} U_1^{-1} & -U_1^{-1}G_1 & U_1^{-1}G_1G_2 & -U_1^{-1}G_1G_2G_3 \\ 0 & U_2^{-1} & -U_2^{-1}G_2 & U_2^{-1}G_2G_3 \\ 0 & 0 & U_3^{-1} & -U_3^{-1}G_3 \\ 0 & 0 & 0 & U_4^{-1} \end{bmatrix} . \tag{20}$$

In Lemma 1, we have shown that the matrix norms of L_i, $i = 2, \ldots, n$ are bounded from above by quantities that are less than unity. On the other hand, Lemma 2 shows that the subdiagonal block of $\mathcal{L}^{-1}$ are composed of block elements which are products of L_i. Thus the matrix norm of the subdiagonal block elements in the j^{th} subdiagonal block are smaller than that of the i^{th} subdiagonal block when $j < i$. This means that the significance of the block elements depends only on its relative distance measured from the main diagonal block.

We need another lemma to characterize the matrix norms of the blocks in $\mathcal{U}^{-1}$, particularly, the matrices G_j, $j = 1, \ldots, n - 1$.

Lemma 3:

The matrix norms of G_j are bounded from above by

$$\|G_j\| \leq \tilde{a}_j \quad , \quad j = 1, \ldots, n-1 \tag{21}$$

where

$$\tilde{a}_j = \frac{a_{j+1}}{1 - \tilde{b}_j a_{j+1}} \quad . \tag{22}$$

The proof of this lemma is given in the Appendix.

This lemma shows that $\mathcal{U}^{-1}$ is block diagonal dominant. The significance of the superdiagonal block elements diminishes for those block elements which are further away from the main diagonal block. This property is to be expected since $\mathcal{L}^{-1}$ has the same property and the inversion of $\mathcal{U}$ and $\mathcal{L}$ are essentially carried out using Lemma 2.

The end purpose of studying the properties of $\mathcal{U}$ and $\mathcal{L}$ is to characterize M^{-1} and ultimately the approximation of $M^{-1}K$ which is the key to arriving at approximate models for (1). The next lemma which results from direct matrix multiplication of $\mathcal{U}^{-1}$ and $\mathcal{L}^{-1}$ shows that the block diagonal dominance of M allows M^{-1} to be approximated a block tridiagonal matrix.

Theorem 1:

Suppose the mass matrix M in (2) is block diagonal dominant, *i.e.*, the inequality (10) holds, then

$$M^{-1} = \operatorname{diag}\left(U_1^{-1}, U_2^{-1}, \ldots, U_n^{-1}\right) \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & & \vdots \\ \vdots & & \ddots & \\ N_{2n} & \cdots & & N_{nn} \end{bmatrix} , \tag{23}$$

where the N_{ij} submatrix is given by:

for $i > j,\ i = j + k$,

$$N_{ij} = (-1)^k L_i L_{i-1} \dots L_{j+1} + (-1)^{k+2} G_i L_{i+1} L_i \dots L_{j+1} + \\ + (-1)^{k+4} G_i G_{i+1} L_{i+2} L_{i+1} \dots L_{j+1} + \dots + \\ + (-1)^{i+j-2} G_1 G_2 \dots G_{n-1} L_n L_{n-1} \dots L_{j+1} \quad , \tag{24}$$

for $i < j,\ j = i + \ell$

$$N_{ij} = (-1)^{\ell} G_i G_{i+1} \dots G_{i+\ell-1} + (-1)^{\ell+2} G_i G_{i+1} \dots G_{i+\ell} L_{j+1} + \\ + (-1)^{\ell+4} G_i G_{i+1} \dots G_{i+\ell+1} L_{j+2} L_{j+1} + \dots + \\ + (-1)^{i+j-2} G_1 G_2 \dots G_{n-1} L_n L_{n-1} \dots L_{j+1} \quad , \tag{25}$$

and for $i = j$

$$N_{ii} = I + G_i L_{i+1} + G_i G_{i+1} L_{i+2} L_{i+1} + \dots + \\ + G_1 G_2 \dots G_{n-1} L_n L_{n-1} \dots L_{i+1} \quad , \tag{26}$$

with $L_p = 0,\ p \in \{n+1, n+2, \dots\}$, $L_r = 0,\ r \in \{1, 0, -1, -2, \dots\}$ and $G_q = 0,\ q \in \{n, n+1, \dots\}$ Moreover, if the matrix norms defined in (8) and (9) are bounded from above by positive scalars proportional to a positive constant $\epsilon < 1$ such that

$$a_i \le \alpha_i \epsilon < 1 \quad , \quad b_i \le \beta_j \epsilon < 1 \quad , \quad \text{for } i = 2, \dots, n \text{ and } j = 1, \dots, n-1 \quad , \tag{27}$$

then M^{-1} can be approximated by a tridiagonal matrix to $O(\epsilon^2)$*, *i.e.*,

$$M^{-1} = \underbrace{\begin{bmatrix} U_1^{-1} & -U_1^{-1} G_1 & 0 & \dots & 0 \\ -U_2^{-1} G_2 & U_2^{-1} & -U_2^{-1} G_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -U_{n-1}^{-1} G_{n-1} \\ 0 & \dots & 0 & -U_n^{-1} L_n & U_n^{-1} \end{bmatrix}}_{S_0} + 0(\epsilon^2) \tag{28}$$

Proof of this theorem is given in the Appendix. Other properties of M^{-1} which are useful in constructing approximation models can be further deduced from the proof, *e.g.*, the upperbounds for the matrix norms of G_j and L_i in the matrix S_0 are proportional to ϵ. Hence, S_0 can be further approximated by

$$S_0 = U_D^{-1} + O(\epsilon) \quad , \tag{29}$$

where

$$\mathcal{U}_D^{-1} = \text{diag}(U_1^{-1}, \dots, U_2^{-1}, \dots, U_n^{-1}) \quad . \tag{30}$$

* The matrix $W(\mu)$ is said to be of order μ, $(O(\mu))$ if there exists positive constants c and μ' such that the matrix norm of W satisfies $\|W(\mu)\| < c\mu$ for all $0 < \mu \le \mu'$.

Other block tridiagonal matrix approximations forM^{-1} can be derived from (23) using the expressions for N_{ij}. For example, instead of utilizing upperbounds for a_i's and b_j's which are proportional to a constant ϵ, the actual bounds given in (8) and (9) can be used in another approximation.

Theorem 1 allows the finite element models of the type of (1), in which the mass matrix is block diagonal dominant, to be approximated. Approximations of the matrix $M^{-1}K$ which dictates the homogeneous solution of (4) are obtained from using the approximation for M^{-1} in (28).

Lemma 4:

Let $\Omega := S_0K$, then Ω is a block quadridiagonal matrix. It's main diagonal block is

$$\begin{aligned} &U_1^{-1}(A_1 - G_1B_2), U_2^{-1}(A_2 - L_2C_1 - G_2B_3), \ldots, \\ &\ldots, U_k^{-1}(A_k - L_kC_{k-1} - G_kB_{k+1}), \ldots, \\ &\ldots, U_{n-1}^{-1}(A_{n-1} - L_{n-1}C_{n-2} - G_{n-1}B_n), U_n^{-1}(A_n - L_nC_{n-1}) . \end{aligned} \tag{31}$$

There are two superdiagonal blocks, the first and the second ones are prescribed by the following equations respectively:

$$\begin{aligned} &U_1^{-1}(C_1 - G_1A_2), \; U_2^{-1}(C_2 - G_2A_3), \ldots, \\ &\ldots, U_{n-1}^{-1}(C_{n-1} - G_{n-1}A_n) \;, \end{aligned} \tag{32a}$$

$$-U_1^{-1}G_1C_2, \; -U_1^{-1}G_2C_3, \ldots, -U_{n-2}^{-1}G_{n-2}C_{n-1} \;, \tag{32b}$$

the first and the second subdiagonal blocks are given in these two equations respectively:

$$\begin{aligned} &U_2^{-1}(-L_2A_1B_2), \; U_3^{-1}(-L_3A_2B_3), \ldots, \\ &\ldots, U_n^{-1}(-L_nA_{n-1} + B_n) \quad , \end{aligned} \tag{33a}$$

$$-U_3^{-1}L_3B_2, \; -U_4^{-1}L_4B_3, \ldots, -U_n^{-1}L_nB_{n-1} \; . \tag{33b}$$

The matrix $M^{-1}K$ is approximated to $O(\epsilon^2)$ by Ω,

$$M^{-1}K = \Omega + O(\epsilon^2) \quad . \tag{34}$$

Furthermore, $M^{-1}K$ is approximated to $O(\epsilon)$ by a block tridiagonal matrix,

$$M^{-1}K = \mathcal{U}_D^{-1}K + O(\epsilon) \quad . \tag{35}$$

The proof of this lemma directly follows the observation that the multiplication of two block tridiagonal matrices results in a block quadridiagonal matrix, and that the off main diagonal blocks of $M^{-1}K$ are $O(\epsilon)$. This Lemma is useful in approximating the solution of (4).

Application to Active Structural Control

The approximate finite element models developed in the last section allow the designers to adhere to the Rayleigh-Ritz procedure for deriving the discretized structural models and yet be able to harvest from the simplicity of having a block diagonal mass matrix structure. As it is shown in Theorem 1, these approximations are possible if the mass matrix is block diagonal dominant. In the design of active structural control, these approximate finite element models are particularly useful in decomposing the control law and the design into independent problems.

For the ensuing discussions, active structural control refers to the use of generalized forces in changing the mode shapes and frequencies of the structure. Realistically these forces should be applied at only a selective number of nodes. For this purpose, a control vector u is introduced, each component of this vector acts on a block of the nodes and these nodal blocks are defined by the block partitions adopted in the mass and stiffness matrices, *i.e.*

$$f_i = H_i u_i \quad , \quad i = 1, \ldots, n \quad . \tag{36}$$

The open loop structure dynamic equation (4) can then be rewritten as

$$\ddot{x} = M^{-1}Kx + M^{-1}Hu, \tag{37a}$$

where

$$H := \mathrm{diag}(H_1, \ldots, H_n)$$

and

$$u^T := (u_1, \ldots, u_n) \quad . \tag{37b}$$

The matrices H_i thus reflect the locations at which the control forces are to be applied. Let dim $u_i = m_i$. If dim $x_i = n_i$, then $m_i \le n_i$. Those rows of H_i corresponding to the nodes on which no control force is to be applied can be set to zero.

A control law which produces a generalized force vector of the following form is considered.

$$f = D_f \dot{x} + K_f x \quad , \tag{38a}$$

where D_f and K_f are block tridiagonal matrices,

$$D_f = \begin{bmatrix} V_1 & T_1 & 0 & \ldots & 0 \\ S_2 & V_2 & T_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & & & T_{n-1} \\ 0 & \ldots & 0 & S_{n-1} & V_n \end{bmatrix} ,$$

$$K_f = \begin{bmatrix} A_1^f & C_1^f & 0 & \ldots & 0 \\ B_1^f & A_2^f & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & C_{n-1}^f \\ 0 & \ldots & 0 & B_{n-1}^f & A_n^f \end{bmatrix} . \tag{38b}$$

The motivation for seeking such a control law is derived from the desire of generating active control forces which are compatible with the internal restoring forces of the structure. This compatibility is enforced by constraining the D_f and the K_f matrices to be block tridiagonal matrices.

The problem of designing such a control law which stabilizes the structure is approached with the help of the developed approximate finite element models. Let the matrices H_i be chosen such that for each eigenvalue

$$\sigma \in \lambda \begin{pmatrix} 0 & I \\ -U_i^{-1}A_i & 0 \end{pmatrix} \quad , \tag{39}$$

$$\text{rank } \left(\sigma^2 U_i + A_i H_i\right) = n_i \quad , \quad \text{for } i = 1, \ldots, n \quad . \tag{40}$$

This condition essentially guarantees that the system

$$\ddot{w} = -U_D^{-1} K w + U_D^{-1} H \nu^c \quad , \tag{41}$$

is controllable. For each of the system, *i.e.*, for 1, ..., n,

$$\dot{\eta}_{i1} = U_i^{-1} \eta_{i2} \quad , \quad \eta_{i1}(0) = x_i(0) \tag{42a}$$

$$\dot{\eta}_{i2} = -A_i \eta_{i1} + H_i \nu_i, \; \eta_{i2}(0) = U_i \dot{x}_i(0) \ , \tag{42b}$$

design an optimal linear feedback control law to minimize a quadratic performance index

$$J_i \left(x_i(0),\ U_i \dot{x}_i(0)\right) = \frac{1}{2} \int_0^\infty \left[\begin{pmatrix} \eta_{i1} \\ \eta_{i2} \end{pmatrix}^T Q_i \begin{pmatrix} \eta_{i1} \\ \eta_{i2} \end{pmatrix} + \nu_i^T R_i \nu_i \right] \mathrm{dt} \ , \tag{43}$$

with positive definite Q and R matrices. The optimal control law for ν_i is

$$\nu_i = -R_i^{-1} \begin{pmatrix} 0 & H_i^T \end{pmatrix} P_i \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \end{bmatrix} \quad , \tag{44}$$

where P_i is the solution of a Riccati equation

$$\begin{aligned} & P_i \begin{pmatrix} 0 & U_i^{-1} \\ -A_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & U_i^{-1} \\ -A_i & 0 \end{pmatrix}^T P_i \\ & -P_i \begin{pmatrix} 0 \\ H_i \end{pmatrix} R_i^{-1} \begin{pmatrix} 0 & H_i^T \end{pmatrix} P_i + Q_i = 0 \end{aligned} \quad . \tag{45}$$

A control law which produces generalized forces in the form of (38) and stabilizes the structure modeled by (37) can then be constructed using the optimal feedback gains in (44): let the submatrices in D_f and K_f in (38) be chosen according to the following equations,

$$\begin{bmatrix} A_i^f & V_i \end{bmatrix} = H_i R_i^{-1} \begin{pmatrix} 0 & H_i^T \end{pmatrix} P_i \begin{bmatrix} I & 0 \\ 0 & U_i \end{bmatrix} \tag{46a}$$

$$C_i^f = -H_i \left(H_i^T H_i\right)^{-1} H_i^T C_i \tag{46b}$$

$$B_i^f = -H_i \left(H_i^T H_i\right)^{-1} H_i^T B_i \tag{46c}$$

$$T_i = S_i = 0 \quad . \tag{46d}$$

The design procedure of the active control law involves the equations (61), (62), and (65). The block diagonal gain matrices A_i^f and V_i are derived from a set of independent optimal linear quadratic optimal regulator problems which are formulated with respect to an approximated model of the actual system. Off-block diagonal coupling terms in this model are removed, allowing the optimal regulator problems to be solved independently. The off-block diagonal gain matrices C_i^f and B_i^f are chosen to minimize the sum of matrix norms,

$$\sum_{i=1}^{n} \|\bar{H}_i C_i\|_2 + \|\bar{H}_i B_i\|_2 \tag{47a}$$

where

$$\bar{H}_i := I - H_i \hat{H}_i \quad , \tag{47b}$$

$$\hat{H}_i = \left(H_i^T H_i\right)^{-1} H_i^T \quad . \tag{47c}$$

If the matrix H_i has the property that

$$\text{rank}\,[H_i \; C_i] = \text{rank}\,[H_i \; B_i] = \text{rank}\; H_i \quad , \tag{48}$$

then $\bar{H}_i C_i = 0, \bar{H}_i B_i = 0$, and the matrix norm defined in (47a) is zero. This implies that if the feedback gain matrices in (46) are used in the control law

$$H \nu^c = D_f \dot{w} + K_f w \quad , \tag{49}$$

where D_f and K_f are defined in (38b), the performance index of (41) which is defined by $J_w\left(x(0),\ \dot{x}(0)\right)$

$$= \frac{1}{2} \sum_{i=1}^{n} \int_0^\infty \left[\left[w_i^T \;\; (U_i \dot{w}_i)^T \right] Q_i \begin{bmatrix} w_i \\ U_i \dot{w}_i \end{bmatrix} + \nu_i^{cT} R_i \nu_i^{cw} \right] \text{dt} \; , \tag{50}$$

attains it minimum value

$$J_w^* \triangleq \min_{\nu^c} \; J_w\left(x(0),\ \dot{x}(0)\right) = \sum_{i=1}^{n} J_i^*\left(x_i(0),\ U_i \dot{x}_i(0)\right) \quad . \tag{51}$$

The next theorem establishes a sufficient condition under which the the performance of the closed loop, actively controlled structure, using the controller gains defined in (65), can be predicted from the performance of the approximate finite element model (41) under the feedback control actions defined in (44). Let the performance index of the structure be chosen as

$$J\left(x(0),\ \dot{x}(0)\right) = \frac{1}{2} \sum_{i=1}^{n} \int_0^\infty \left[\left(x_i^T \; \dot{x}_i^T \right) Q_i \begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} + u_i^T R_i u_i \right] \text{dt} \; , \tag{52}$$

Let

$$A_{fw} \triangleq \begin{bmatrix} A_{fw}^1 & \dots & A_{fw}^{C,1} & 0 & \dots & 0 \\ A_{fw}^{B,2} & \ddots & A_{fw}^2 & \ddots & \ddots & \vdots \\ 0 & \ddots & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & \ddots & A_{fw}^{C,n-1} \\ 0 & \dots & 0 & A_{fw}^{B,n} & \dots & A_{fw}^n \end{bmatrix} \tag{53a}$$

$$A_{fw}^i \triangleq \begin{bmatrix} 0 & U_i^{-1} \\ A_i & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ H_i \end{bmatrix} K_D^i \;,\; A_{fw}^{C,i} = \begin{bmatrix} 0 & 0 \\ \bar{H}_i C_i & 0 \end{bmatrix} \; A_{fw}^{B,i} = \begin{bmatrix} 0 & 0 \\ \bar{H}_i B_i & 0 \end{bmatrix} \tag{53b}$$

$$K_{fw} \triangleq \begin{bmatrix} K_D^1 & K_C^1 & 0 & & \cdots & 0 \\ K_B^2 & K_D^2 & K_C^2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & 0 \\ \vdots & \ddots & & & \ddots & K_C^{n-1} \\ 0 & \cdots & 0 & & K_B^n & K_D^n \end{bmatrix} , \tag{53c}$$

$$K_D^i \triangleq -R_i^{-1} \left(0 \; H_i^T \right) P_i \quad K_C^i \triangleq - \left(\hat{H}_i C_i \; 0 \right) \quad K_B^i \triangleq - \left(\hat{H}_i B_i \; 0 \right) \tag{53d}$$

Theorem 2:

Let

$$A_{cw} \triangleq A_{fw} - \operatorname{diag}\left(A_{fw}^1, \ldots, A_{fw}^n \right) \quad , \tag{54a}$$

$$K_D \triangleq \operatorname{diag}\left(K_D^1, \ldots, K_D^n \right) \quad K_G \triangleq K_{fw} - K_D \quad , \tag{54b}$$

$$R_D \triangleq \operatorname{diag}\left(R_1, \ldots, R_n \right) \; Q_D \triangleq \operatorname{diag}\left(Q_1, \ldots, Q_n \right) \; . \tag{54c}$$

An aggregate matrix W_g of the closed loop system (41), (50) is an $n \times n$ matrix whose elements $(W_g)_{ij}$ are given by

$$(W_g)_{ij} = \begin{cases} -(1-\mu)\, \lambda_M \left(Q_i + \left(K_D^i\right)^T R_i K_D^i \right) \quad , & i = j \; . \\ \lambda_M^{\frac{1}{2}} \left(\left[(A_{cw})^T P_D + P_D A_{cw} + \mu \left(K_D + K_G\right)^T R_D \left(K_D + K_G\right) \right]_{ij} \right) & i \neq j \end{cases} \tag{55}$$

where $[T]_{ij}$ denotes the ij block submatrix of a block matrix T.

If for some positive number μ, W_g is quasidominantdiagonal, then the performance index of the approximate finite element model (41) with the control law (44) is bounded from above by

$$J_w \leq \mu^{-1} \sum_{i=1}^{n} J_i^* \left(x_i(0), U_i \dot{x}_i(0) \right) \quad . \tag{56}$$

Moreover, the performance of the structure with the control law defined by the equations (38b) and (46) is bounded by

$$J \left(x(0), \; \dot{x}(0) \right) \leq J_w + 0(\epsilon^2) \quad . \tag{57}$$

Illustrative Examples:

Linear matrix-second-order systems resulting from an application of a finite element method to solve structural dynamics problems are used herein as numerical examples. Specifically, a plane truss vibration problem is considered. This truss structure is depicted in Figure 1. The assumptions made are that the truss members are subjected to axial forces alone, and not bending moments; and the members are uniform rods of identical lengths L, mass per unit length m, cross-section area per unit length A and modulus of elasticity E. This truss has five bays and the nodal coordinates are defined as the vertical and horizontal displacements at the joints. External forces applied at the nodes are decomposed into orthogonal components. The mass and stiffness matrices of the truss are derived using a finite element method with the Ritz-Rayleigh approximation. The element mass and stiffness matrices used in the assembly process are:

$$k = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad m = \frac{mL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad . \tag{58}$$

The assembled mass and stiffness matrices are banded matrices with an identical structure as illustrated in Figure 2. In this diagram, the nonzero elements are indicated by "*". These matrices are scaled to remove the effects of the material properties, that is, a new time variable τ is introduced where $\tau = (\frac{m}{6EA})^{\frac{1}{2}} L$ and the nodal forces are scaled by $\frac{L}{EA}$.

The mass and stiffness matrices can be block-partitioned into a block tridiagonal matrices as in (2) and (3). Consider a possibility of $n = 6$ and dim $x_i = 4$, for all i. Direct computation shows that M is not explicit block diagonal dominant, but it is implicit block diagonal dominant. The matrix norms as defined in (8) and (9) are:

$$a_2 = a_3 = a_4 = a_5 = 0.4086 \quad , \tag{59}$$

$$a_6 = 0.5663 \quad , \tag{60}$$

$$b_1 = 0.5663 \quad , \tag{61}$$

$$b_2 = b_3 = b_4 = b_5 = 0.4086 \quad , \tag{62}$$

According to Theorem 1, we compute the matrix S_0 which is an $O(\epsilon^2)$ approximation to M^{-1}, and the matrix U_D^{-1} which is an $O(\epsilon)$ approximation. The norm of the difference between M^{-1} and its approximations are found to be:

$$\|M^{-1} - S_0\| = 0.0953 \tag{63}$$

and

$$\|M^{-1} - U_D^{-1}\| = 0.2378 \quad . \tag{64}$$

The approximations of the matrix $M^{-1}K$ as given in Lemma 4, on the other hand, give these norms of difference:

$$\|M^{-1}K - S_0K\| = 0.3027 \tag{65}$$

and

$$\|M^{-1}K - U_d^{-1}K\| = 0.8417 \quad . \tag{66}$$

The square roots of the eigenvalues of $M^{-1}K$, S_0K and $U_D^{-1}K$ are tabulated in Table 1 for comparison. They represent the modal frequencies of the unforced system and that of the two approximate models. As expected, the three zero eigenvalues representing the rigid body modes are preserved in the approximate finite element models.

A control design using the design procedure outlined earlier in this paper was carried out. In this design, the mass and stiffness matrices are block-partitioned such that $n = 3$ and dim $x_i = 8$ for $i = 1, \ldots, n$. The number of control variables per block partition is chosen to be 4, *i.e.*, dim $u_i = 4$. The matrix H_i which reflects where the control forces are to be applied with respect to the nodes is chosen as

$$H_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad i = 1, \ldots, 3 \tag{67}$$

The objective used in selecting this matrix is primarily to insure that the system (42) is controllable. The quadratic weighting matrices, Q_i and R_i in (43) are chosen to be identity matrices. Three sixteenth order Riccati equations were solved and the controller gains are computed accordingly to (46). Figure 3 which shows the eigenvalues of the actual closed loop system and that of the approximate finite element model subject to the designed control law. Figure 4 displays some sample trajectories of the actual closed loop system, indicating that structural vibrations are successfully damped out with the use of active structural control.

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Appendix

Proofs of the Theorems and Lemmas are omitted herein due to space limitations. A complete version of this paper is available from the author.

Table 1. Modal Frequency Comparison

	$n = 1$ to 8							
$\sqrt{\lambda}(M^{-1}K)$	1.4142	1.3448	1.3132	1.2910	1.2712	1.1930	1.1882	1.1601
$\sqrt{\lambda}(S_0K)$	1.3426	1.3082	1.2719	1.2564	1.2175	1.1812	1.1658	1.1413
$\sqrt{\lambda}(U_D^{-1}K)$	1.2729	1.2541	1.1582	1.1359	1.1103	1.1084	0.9866	0.9512

	$n = 9$ to 16							
$\sqrt{\lambda}(M^{-1}K)$	0.9594	0.8214	0.7053	0.7051	0.5853	0.5020	0.3934	0.3861
$\sqrt{\lambda}(S_0K)$	0.9589	0.8276	0.7014	0.6863	0.5900	0.4994	0.3838	0.3806
$\sqrt{\lambda}(U_D^{-1}K)$	0.8614	0.7968	0.7453	0.7332	0.6108	0.5610	0.4377	0.3775

	$n = 17$ to 24							
$\sqrt{\lambda}(M^{-1}K)$	0.3150	0.3020	0.2283	0.2034	0.1120	0.0000	0.0000	0.0000
$\sqrt{\lambda}(S_0K)$	0.2971	0.2962	0.2110	0.1973	0.1098	0.0000	0.0000	0.0000
$\sqrt{\lambda}(U_D^{-1}K)$	0.3534	0.3057	0.2774	0.2164	0.1197	0.0000	0.0000	0.0000

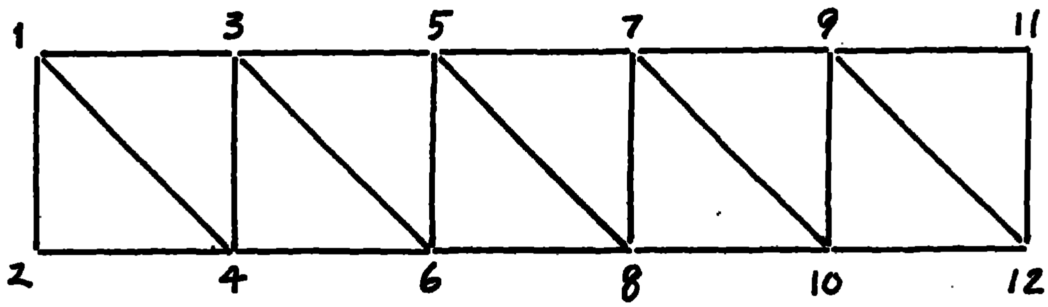

Figure 1. A five bay truss structure with node numbers indicated.

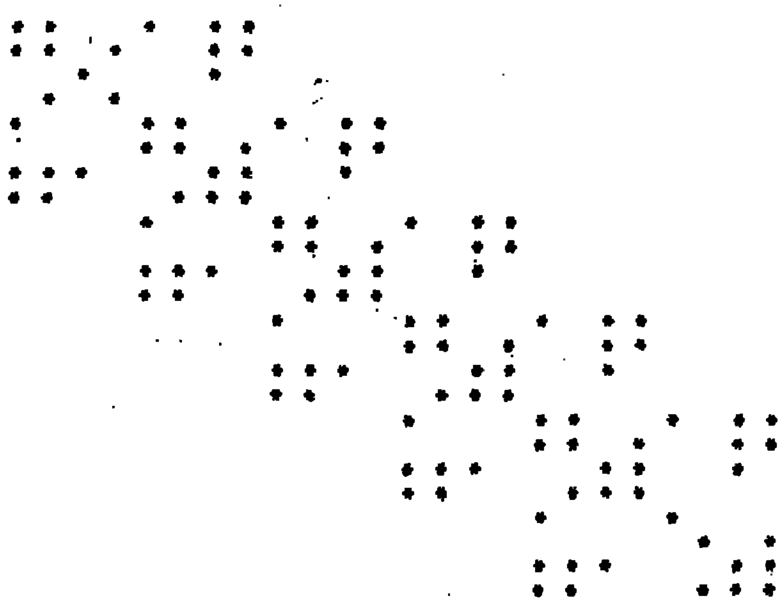

Figure 2. Structure of the mass and stiffness matrix.

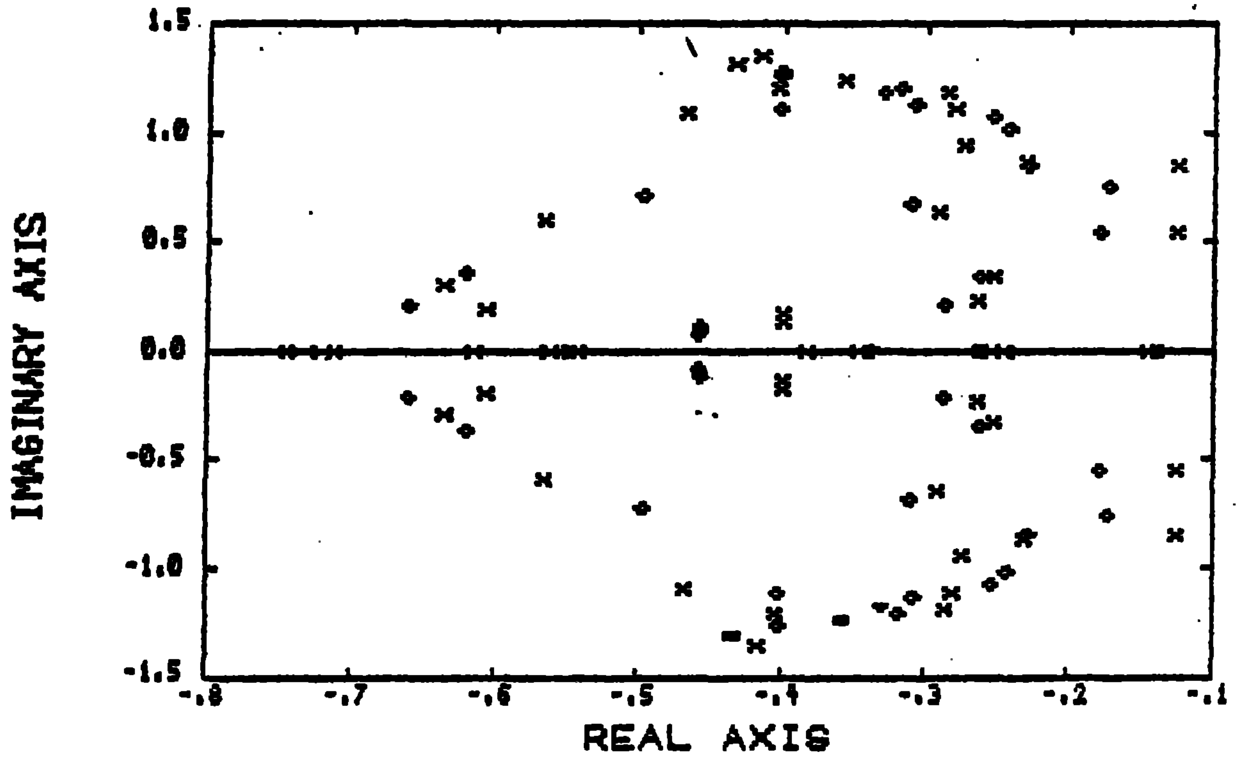

Figure 3. Close loop system eigenvalues: x approximate system and ◇ actual system.

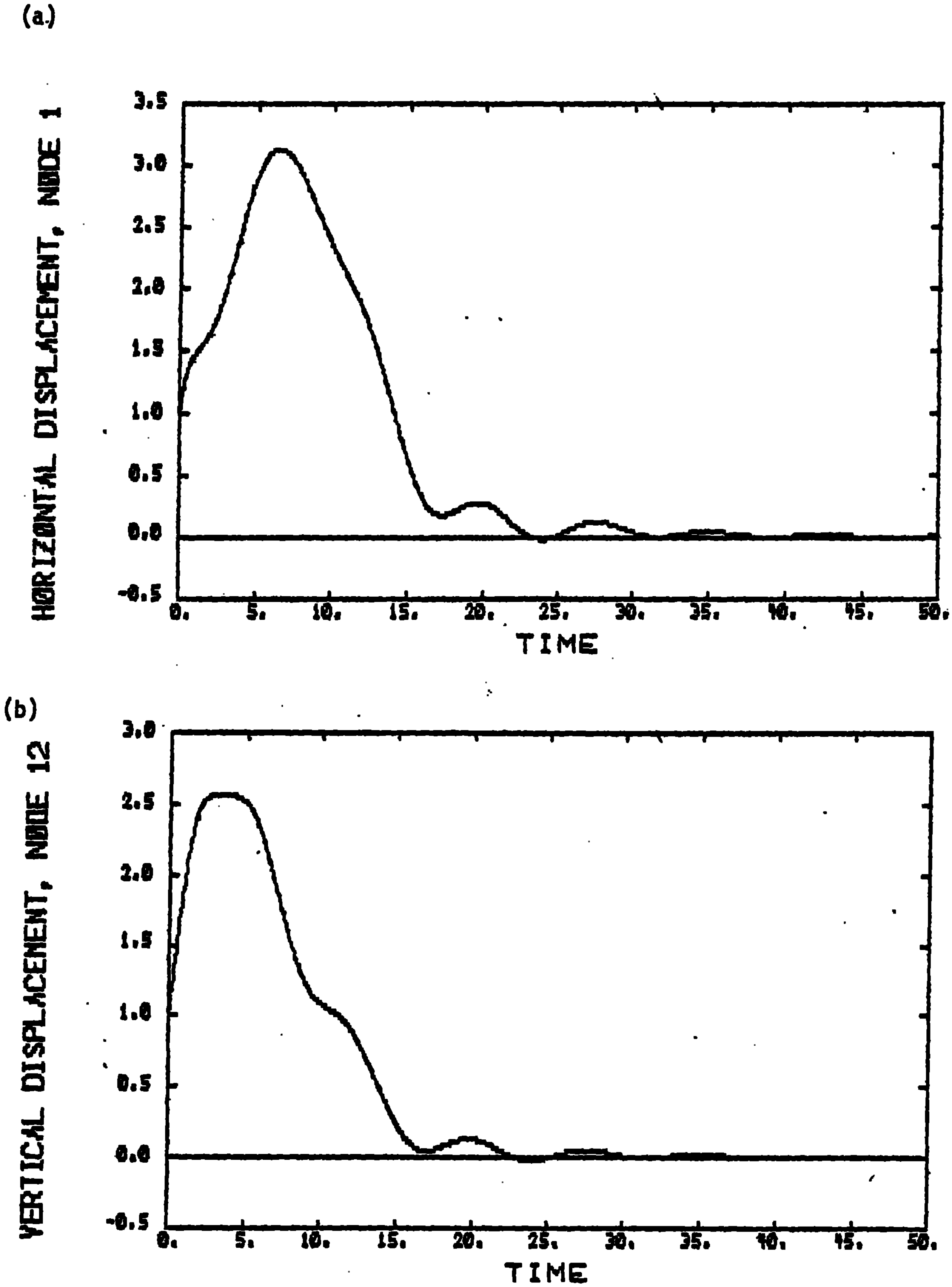

Figure 4(a), (b). Closed loop system respones due to inital conditions, $x_i(0) = \dot{x}_i(0) = 1,\ j = 1, \ldots, 24.$